A strictly increasing continuous function f on $[0,1]$ might have derivative zero almost everywhere in the sense of the Lebesgue measure.

Question: Let f be a strictly increasing continuous function from $[0,1]$ to $\mathbb{R}$. From a well-known result in real analysis, f is differentiable almost everywhere. Let $E = \{x \in [0,1] \colon f'(x) = 0\}$. As f is striclty increasing, using μ to denote the Lebesgue measure, can we claim that $\mu(E) = 0$?

It originates from a question from the class. Some related facts and backgrounds will be given here, for your reference.

Let f be an increasing function (not required to be strictly increasing) on interval $[a,b]$. Surely we shall not expect f to be differentiable everywhere on $[a,b]$. However, as an important and classical result which is not trivial at all in real analysis (Lebesgue's Theorem: every monotone functoin on $[a,b]$ is differentiable a.e. in the sense of the Lebesgue measure), we know that f must be differentiable almost everywhere in the sense of the Lebesgue measure μ.

Use D to denote the points at which this f is differentiable. Then $\mu(D) = \mu([a,b]) = b - a$ and $\mu([a,b] - D) = 0$.

A natural question is, shall we still expect to have such things like the Newton-Leibniz formula

$$\int_D f' \, \mathrm{d}\mu = f(b) - f(a),$$

where the left hand side is the Lebesgue integration instead of the classical Riemann integration?

In short, **no!**

In fact, you guys have already seen such an example in your homework, the Cantor function. Regard the Cantor set X as a subset of $[0,1]$, which is constructed by keep taking off the open one-thirds in the midle. Then the Cantor function $\varphi \colon [0,1] \to [0,1]$ is an increasing (not strictly increasing though) function which has derivative zero on $[0,1] - X$. In other words, $\varphi' = 0$ a.e. in the sense of the Lebesgue measure μ. Note that $\varphi(1) = 1$ and $\varphi(0) = 0$. We have

$$0 = \int_{[0,1]-X} \varphi' \, \mathrm{d}\mu \lneqq \varphi(1) - \varphi(0) = 1.$$

With that in mind, one should not expect to have Newton-Leibniz alike formulas for Lebesgue integrations of derivatives on differentiable subsets.

Now, back to the original question.

We will construct a continuous strictly increasing function f on $[0,1]$, such that $f' = 0$ a.e. in the sense of the Lebesgue measure. This construction process is somewhat similar to the construction process of the above mentioned Cantor function.

Will fill in details of this when time allows...

Remark: For the continuous strictly increasing function f on $[0,1]$, if we furthur assume that f is differentiable and f' is continuous on $[0,1]$, or equivalently, $f \in C^1[a,b]$, then we can prove (need some work though) that $\mu(\{x\colon f'(x) = 0\}) = 0$, where μ is the Lebesgue measure. This shows that differentiation structure ($C^1, C^2, \cdots, C^k, C^\infty$, etc) can rule out some phenomenon that is against our naive intuition.